J. Fluid Mech. (1983), *vol.* 129, *pp.* 265–281
Printed in Great Britain

Viscoelastic squeeze-film flows – Maxwell fluids

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(Received 14 May 1982 and in revised form 12 November 1982)

An exact solution for the squeeze-film motion in an upper convected Maxwell fluid is given for both the plane and axisymmetric cases. Inertia and viscoelastic effects are included, and it is shown that the solution depends only on the product of the Weissenberg and Reynolds numbers. Solutions are generated for values of this product up to 500 without numerical problems. The solutions show wave propagation and show a reduced load capacity relative to the Newtonian case.

1. Introduction

We consider two parallel plates approaching or receding from one another (figure 1a). The gap between the plates is filled with fluid, and we seek to calculate the forces needed to maintain the motion. Although the plates may have any planform, we shall only deal with the axisymmetric case (circular plates of radius a) and the plane case (long two-dimensional strip plates of width $2L$). The flow is of considerable interest in that it models the action of a lubricant in a bearing under unsteady load conditions and is also relevant to the interpretation of various plastometer measurements.

The mechanics of a thin film of fluid being squeezed between two parallel plates has a long history, dated back to Stefan (1874) and Reynolds (1886). The flow forms the basic configuration of a commercial viscometer known variously as the 'parallel-plate viscometer' (Diennes & Klemm 1946), the 'compression plastometer' (Mooney 1958), the 'parallel-plate plastimeter' (Scott 1931, 1932) or simply the 'plastometer' as it is known today. Early workers in this field were primarily concerned with Newtonian fluids in creeping flow (Stefan 1874; Reynolds 1886). In the works of Jackson (1962), Kuzma (1967), Jones & Wilson (1975), Grimm (1976), MacDonald (1977) and the recent work of Hamza & MacDonald (1981) inertia was included in the Newtonian flow calculation. However, many common lubricants are non-Newtonian, and certainly most substances tested in plastometers are not Newtonian; hence the investigations of the flow need to be extended in this direction. Early investigations of inelastic non-Newtonian fluids were made by Scott (1931, 1932) using a power-law model and the traditional assumptions of lubrication kinematics. It is believed that these results are valid for this class of fluids for small (gap/plate-radius) values. The problems of viscoelastic non-Newtonian fluids in inertialess flow were investigated by Tanner (1965), who assumed a type of nonlinear Maxwell model, and more recently by Kramer (1974), who used a different Maxwell model (the Lodge rubber-like fluid). Tanner (1965) assumed that the flow is locally an unsteady shear flow, but in fact the flow is a mixture of stretching and shearing, and is kinematically complex. It is possible to perform an analysis for the linear viscoelastic case, but neither this analysis not the analysis of Tanner (1965) really explains the discrepancy between such theories, all of which show a *reduction* in the force needed to push the plates together (relative to the inelastic cases analysed by Scott), and the enhancement

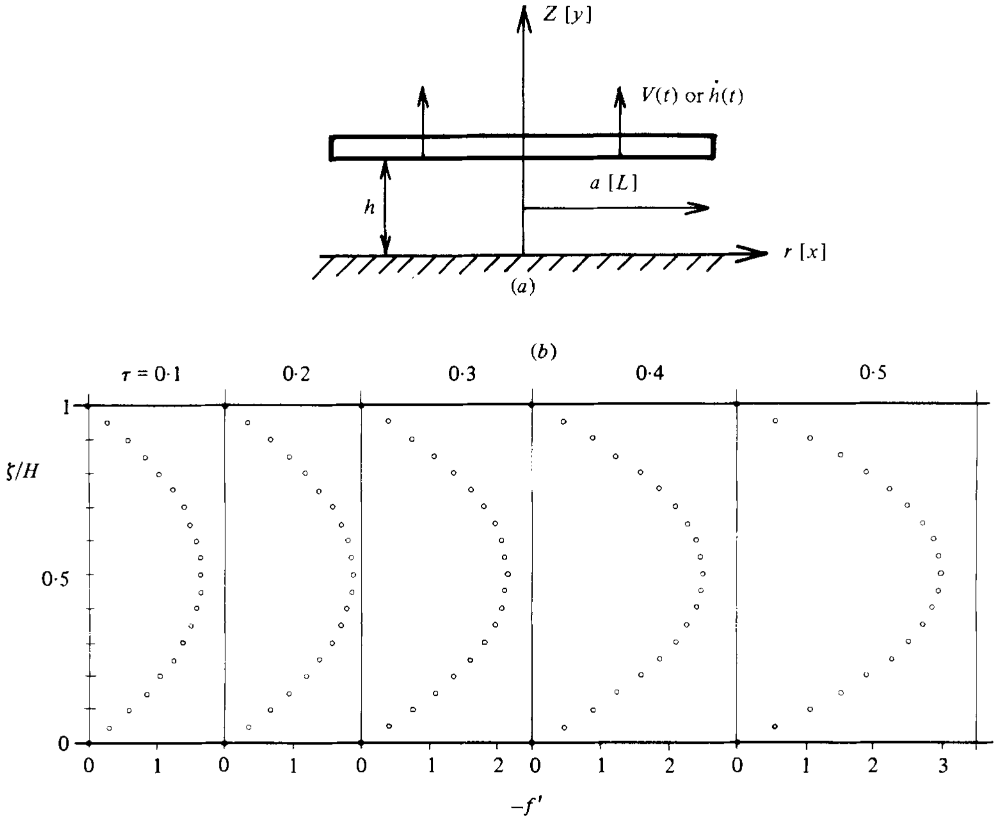

FIGURE 1. (*a*) Coordinate system for problems. [x], [y], [L] refer to plane case, r, z, a to axisymmetric case. (*b*) Horizontal velocity $u/\frac{1}{2}rV$ at $Re = Wi = 0{\cdot}01$ at different times τ. The classical lubrication approximation is established from the onset of the motion.

of load found experimentally by Leider (1974) and Grimm (1978). It was suggested by Metzner (1968) that all the observed increases in load arose from the stretching motion in the fluid; this factor was ignored by Tanner (1965) and Kramer (1974). To perform a rough check on this idea we can assume that the fluid does not stick to the plates. Then a simple analysis of the resulting unsteady shear-free flow shows little effect to assist in interpreting the experiments; some load enhancement is predicted, but it is only of the order of (gap/radius)^2 and hence is insignificant in most practical cases. Since many of the calculations mentioned are approximate, there is need for a more exact analysis of the squeeze film, and this is the object of the present paper.

In this paper we report some exact (including inertia but neglecting body-force and edge effects) plane and axisymmetric solutions to the squeezing flow of a nonlinear Maxwellian liquid. Perturbation solutions are also developed up to terms of first order in the Weissenberg and the Reynolds numbers. Full numerical solutions are reported for the special case where the squeezing velocity (velocity of approach of the plates) is constant.

2. Formulation of problem

The problem we are concerned with is that of a lubricant contained between two parallel plane surfaces. The bottom plate is fixed and the top plate is set in motion at the instant $t = 0+$. The thickness of the lubricant film at any instant of time t is denoted by $h(t)$; the top plate is then moving with a velocity of dh/dt. The fluid is assumed to be incompressible, so that $\nabla . \mathbf{u} = 0$, and the equation of motion is $\nabla . \boldsymbol{\sigma} = \rho d\mathbf{u}/dt$, where the velocity vector is $\mathbf{u}$; the total stress is $\boldsymbol{\sigma}$, and ρ is the fluid density. Body forces such as gravity are ignored here. The constitutive equation for the fluid is written in terms of the extra stress tensor $\mathbf{S}$, where $\boldsymbol{\sigma} = \mathbf{S} - P\mathbf{I}$, P being the pressure and $\mathbf{I}$ being the unit tensor. The lubricant is assumed to behave like an (upper convected) incompressible Maxwellian liquid. That is, if the velocity gradient is $\mathbf{L}^{\mathrm{T}} = \nabla \mathbf{u}$ ($L_{ij} = \partial u_i/\partial x_j$) and the strain rate is $\mathbf{D} = \frac{1}{2}(\mathbf{L}+\mathbf{L}^{\mathrm{T}})$, where the superscript T denotes a transpose, then the stress $\mathbf{S}$ is given by

$$\mathbf{S} + \lambda(\partial_t \mathbf{S} + \mathbf{u} . \nabla \mathbf{S} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^{\mathrm{T}}) = 2\eta \mathbf{D}, \tag{1}$$

in which λ is the (constant) relaxation time and η is the (constant) viscosity of the liquid. Using a coordinate system with the origin fixed at the centre of the bottom plate ($\{x, y\}$ for the plane case, with y perpendicular to the plates; and $\{r, \theta, z\}$ for the axisymmetric case, with z perpendicular to the plates, figure 1*a*), the relevant boundary conditions for the velocity $\mathbf{u}$ ((u, v) for the plane case, and $(u, 0, w)$ for the axisymmetric case) are:

$$\left.\begin{aligned} u = v = w = 0 \quad (y = z = 0), \\ u = 0, \quad v = w = \frac{dh}{dt} \quad (y = z = h(t)). \end{aligned}\right\} \tag{2}$$

For $t < 0$, the top plate is at rest, $\dot{h} = 0$ and $h = h_0$; for $t \geqslant 0$ the motion at the top plate $h(t)$ is prescribed. The problem is time-dependent and we must also supply initial conditions. We shall assume that the fluid is at rest originally:

$$\mathbf{u}(\mathbf{x}, t) = 0 \quad (-\infty < t < 0, \quad \text{all } \mathbf{x}). \tag{3}$$

We will show that the velocity fields

$$u = -Vx\,\partial_y f(y,t), \quad v = Vf(y,t); \tag{4}$$

$$u = -\tfrac{1}{2}Vr\,\partial_z f(z,t), \quad w = Vf(z,t) \tag{5}$$

constitute exact solutions to the squeezing flow of a Maxwellian fluid, neglecting body forces and edge effects (the plates are assumed to be infinite in extent), in the plane and axisymmetric cases respectively. In (4) and (5) V is a scale velocity, say the magnitude of the initial velocity of the top plate, and f is a function of time and the vertical distance from the bottom plate, as yet undetermined. Note that (4) and (5) satisfy the conservation of mass identically, and f may be regarded as a stream function. The structure of (4) and (5) is that they do not permit a material plane to buckle. For a generalized Newtonian fluid whose viscosity function is not of the power-law type Brindley, Davies & Walters (1976) showed that a material plane will experience buckling. It is not clear that this is a severe defect in the work of Scott (1931), who did assume a non-buckling velocity field, and in the present paper we shall not concern ourselves with this question. In cases where $R/h \gg 1$ it seems likely to be a good approximation to the true field in every case. In the present work we find no need to postulate buckling of the planes at any Weissenberg number. For brevity we will non-dimensionalize y (or z) by h_0, the initial film thickness, and the time by V/h_0, so that

$$\zeta = \frac{y}{h_0} \quad \text{or} \quad \frac{z}{h_0}, \quad \tau = \frac{Vt}{h_0},$$

and denote derivatives with respect to τ by a dot and derivatives with respect to ζ by a prime.

2.1. *Plane squeezing flow*

Assuming (4), one finds that the constitutive equation (1) becomes

$$S_{xx} + Wi\left(\dot{S}_{xx} - xf'\,\partial_x S_{xx} + fS'_{xx} + 2f'S_{xx} + \frac{2x}{h_0}f''S_{xy}\right) = -2\frac{\eta V}{h_0}f', \tag{6}$$

$$S_{xy} + Wi\left(\dot{S}_{xy} - xf'\partial_x S_{xy} + fS'_{xy} + \frac{x}{h_0}f''S_{yy}\right) = -\frac{\eta V}{h_0^2}xf'', \tag{7}$$

$$S_{yy} + Wi\left(\dot{S}_{yy} - xf'\,\partial_x S_{yy} + fS'_{yy} - 2f'S_{yy}\right) = 2\frac{\eta V}{h_0}f', \tag{8}$$

where $Wi = \lambda V/h_0$ is the Weissenberg number. Conservation of linear momentum requires, neglecting body forces (but not inertia),

$$-\frac{\rho V^2}{h_0^2}x(\dot{f}' - f'^2 + ff'') = -\partial_x P + \partial_x S_{xx} + \partial_y S_{xy}, \tag{9}$$

$$\frac{\rho V^2}{h_0}(\dot{f} + ff') = -\partial_y P + \partial_x S_{xy} + \partial_y S_{yy}. \tag{10}$$

It may be shown that (6)–(10) admit a solution of the form

$$S_{xx} = \frac{\eta V}{h_0}X_1(\zeta,\tau) + \frac{\eta V}{h_0^3}x^2X_2(\zeta,\tau), \tag{11}$$

$$S_{xy} = \frac{\eta V}{h_0^2}xT(\zeta,\tau), \tag{12}$$

$$S_{yy} = \frac{\eta V}{h_0}Y(\zeta,\tau), \tag{13}$$

where, from (6)–(8),

$$X_1 + Wi(\dot{X}_1 + fX'_1 + 2f'X_1) = -2f', \tag{14}$$

$$X_2 + Wi(\dot{X}_2 + fX'_2 + 2f''T) = 0, \tag{15}$$

$$T + Wi(\dot{T} - f'T + fT' + f''Y) = -f'', \tag{16}$$

$$Y + Wi(\dot{Y} + fY' - 2f'Y) = 2f'. \tag{17}$$

Furthermore, from (9) and (10) one finds that for compatibility

$$\left.\begin{aligned} P &= \frac{\eta V}{h_0}\left[\frac{1}{2}\left(\frac{x}{h_0}\right)^2 p(\tau) + \text{function of } \zeta\right], \\ p(\tau) &= 2X_2 + T' + Re(\dot{f}' - f'^2 + ff'') \end{aligned}\right\} \tag{18}$$

where the second line holds, and $p(\tau)$ is a function of time only; that is

$$2X'_2 + T'' + Re(\dot{f}'' - f'f'' + ff''') = 0. \tag{19}$$

In (18) and (19) $Re = \rho V h_0/\eta$ is the Reynolds number.

To sum up, the solution to (14)–(17) and (19) subject to the boundary conditions

$$f(0) = f'(0) = 0, \quad f'(H) = 0, \quad f(H) = \dot{H}, \tag{20}$$

and some initial conditions, constitute an exact solution to the plane squeezing flow of a Maxwellian fluid. In (20) we have normalized the film thickness by h_0:

$$H(\tau) = \frac{h(t)}{h_0}, \quad H(0) = 1, \quad \dot{H}(0+) = -1. \tag{21}$$

If the Weissenberg number is identically zero, the governing equations reduce to

$$Re(\dot{f}'' - f'f'' + ff''') = f^{\text{iv}},$$

which was considered by Jones & Wilson (1975), MacDonald (1977) and Hamza & MacDonald (1981). Note that, by rescaling τ and ζ and the stresses, one can make the coefficient of the inertia term in (19) unity, and it is found that the evolution of f is governed by the product $Re\, Wi$. (The stresses are, of course, different. This remark also applies to the axisymmetric case discussed below.) We shall exploit this feature in checking our numerical works.

2.2. *Axisymmetric squeezing flow*

In a similar manner it is found that (5) is an exact solution to the axisymmetric squeezing flow of a Maxwellian fluid. In this case the stresses are given by

$$S_{rr} = \frac{\eta V}{h_0} R_1(\zeta, \tau) + \frac{\eta V}{h_0^3} r^2 R_2(\zeta, \tau),$$

$$S_{\theta\theta} = \frac{\eta V}{h_0} \Theta(\zeta, \tau),$$

$$S_{zz} = \frac{\eta V}{h_0} Z(\zeta, \tau),$$

$$S_{rz} = \frac{\eta V}{h_0^2} rT(\zeta, \tau),$$

in which

$$R_1 + Wi(\dot{R}_1 + fR_1' + f'R_1) = -f', \tag{22}$$

$$R_2 + Wi(\dot{R}_2 + fR_2' + f''T) = 0, \tag{23}$$

$$\Theta + Wi(\dot{\Theta} + f\Theta' + f'\Theta) = -f', \tag{24}$$

$$Z + Wi(\dot{Z} + fZ' - 2f'Z) = 2f', \tag{25}$$

$$T + Wi(\dot{T} - f'T + fT' + \tfrac{1}{2}f''Z) = -\tfrac{1}{2}f''. \tag{26}$$

Compatibility requires that $\Theta = R_1$, which is automatically satisfied if initial conditions on Θ and R_1 are the same, and

$$\left.\begin{aligned} P &= \frac{\eta V}{h_0}\left[\frac{1}{2}\left(\frac{r}{h_0}\right)^2 p(\tau) + \text{function of } \zeta\right], \\ p(\tau) &= 3R_2 + T' + \tfrac{1}{2}Re(\dot{f}' - \tfrac{1}{2}f'^2 + ff'') \end{aligned}\right\} \tag{27}$$

where

is a function of time only, viz.

$$3R_2' + T'' + \tfrac{1}{2}Re(\dot{f}'' + ff''') = 0. \tag{28}$$

The solution of (22), (23), (25), (26) and (28) subject to the boundary conditions (20) is an exact solution to the axisymmetric squeezing flow of a Maxwellian fluid.

2.3. *Normal force on the top plate*

Although the solutions discussed above are exact only for an infinite squeeze-film bearing, we feel that they form an excellent approximation to the fields in a finite squeeze-film bearing where either the bearing length $2L$ or the bearing radius a is very much greater than h_0, the initial film thickness. One is interested in the load-carrying capacity of the bearing, given by

$$W_{\mathrm{T}} = \int_{-L}^{L} (P - S_{yy})\, dx \quad \text{at} \quad y = h(t),$$

or in the axisymmetric case by

$$W_{\mathrm{T}} = \int_{0}^{a} 2\pi r (P - S_{zz})\, dr \quad \text{at} \quad z = h(t).$$

If the above integrals are evaluated at $y = z = 0$ then the loads W_{B} on the bottom plate are obtained; these differ from those on the top plate by the vertical mass acceleration of the fluid in the gap.

In the plane case one notes that

$$P = \frac{\eta V}{2h_0^3} x p(\tau) + \frac{\eta V}{h_0} Q(\zeta, \tau) + P_0,$$

where $p(\tau)$ is given by (18),

$$Q(\zeta, \tau) = \int_0^{\zeta} \{T(\mu, \tau) + Y'(\mu, \tau) - Re(\dot{f}(\mu, \tau) + f(\mu, \tau) f'(\mu, \tau))\}\, d\mu,$$

and P_0 is a constant of integration. To fix P_0 we assume that the radial traction is zero at the edges:

$$\int_0^{h(t)} (-P + S_{xx})\, dy = 0 \quad (x = \pm L),$$

which implies

$$P_0 = \frac{\eta V L^2}{h_0^3} (\overline{X}_2 - \tfrac{1}{2} p) + \frac{\eta V}{h_0} (\overline{X}_1 - \overline{Q}),$$

where the overbar denotes an average over $y \in [0, h(t)]$, viz

$$\overline{Q}(\tau) = \frac{1}{H} \int_0^H Q(\zeta, \tau)\, d\zeta.$$

The load on the top plate can now be found, and one has

$$W_{\mathrm{T}} = -\frac{2\eta V L^3}{3h_0^3} (p(\tau) - 3\overline{X}_2(\tau)) + \frac{2\eta V L}{h_0} (Q(H, \tau) - \overline{Q}(\tau) + \overline{X}_1(\tau) - Y(H, \tau)).$$

Note that, owing to inertia, the load on the top and the bottom plate are different, so that

$$\frac{W_{\mathrm{T}} - W_{\mathrm{B}}}{W_{\mathrm{T}}} = O\left(\frac{h_0^2}{L^2}\right).$$

Furthermore, by neglecting terms $O(h_0^2/L^2)$, the dimensionless load is given by

$$w = W_{\mathrm{T}} \Big/ \frac{2\eta V L^3}{3h_0^3} = -p(\tau) + 3\overline{X}_2(\tau). \tag{29}$$

The same argument can be applied to the axisymmetric case, and one has

$$w = W_{\mathrm{T}}\bigg/\frac{\pi\eta Va^4}{4h_0^3} = -p(\tau)+4\bar{R}_2(\tau), \tag{30}$$

where of course $p(\tau)$ is given by (27) and $\bar{R}_2$ is the z-average of R_2.

2.4. *Initial conditions*

In numerically integrating the governing equations we will consider only the case where the upper plate is moving impulsively at constant velocity V at time $t = O(\dot{H}(\tau) = -1)$. Such motion is only possible when the normal force at $t = 0+$ is infinite (even for finite plates) and has been considered by Jones & Wilson (1975) and Hamza & MacDonald (1981) for a Newtonian fluid. To discover initial conditions relevant to impulsive motion we integrate the governing equations from $\tau = 0$ to $\tau = 0+$ to find that

$$f(\tau, 0+) = -\zeta, \quad \zeta \neq 0, H. \tag{31}$$

Thus, except for two extremely thin boundary layers near the plates, the fluid will move sideways as a plug at any station x. Note that (31) is the inviscid (shear-free) solution.

Assuming the stresses to be continuous at time $\tau = 0$, one can integrate the constitutive equations (14)–(17) and (22)–(26) exactly to obtain the following initial conditions on the stresses for plane flow at $\tau \approx 0+$:

$$X_1 = X_2 = T = Y = 0,$$

and, for axisymmetric flow at $\tau \approx 0+$,

$$R_1 = R_2 = \Theta = Z = T = 0.$$

3. Perturbation solutions

Asymptotic results can be found in the case where both Re and Wi are small, whence one can write

$$f = f_0 + Re f_{10} + Wi f_{01} + O(Re^2, Re\,Wi, Wi^2),$$

and similar expressions for the stresses. The zeroth-order term (for both plane and axisymmetric case) is the classical lubrication approximation:

$$f_0 = \dot{H}\left(\frac{3\zeta^2}{H^2} - \frac{2\zeta^3}{H^3}\right). \tag{32}$$

In the plane squeezing flow, non-Newtonian effects do not alter the velocity field ($f_{01} = 0$), whereas in the axisymmetric case

$$f_{01} = -6\frac{\dot{H}^2}{H}\left(\frac{\zeta^3}{H^3} - \frac{\zeta^4}{H^4} + \frac{2\zeta^5}{5H^5}\right) + \alpha_1\zeta^3 + \alpha_2\zeta^2,$$

where

$$\alpha_1 = \frac{6\dot{H}^2}{5H^4}, \quad \alpha_2 = \frac{6\dot{H}^2}{5H^3}.$$

However, inertia alters both the plane and the axisymmetric velocity fields: in the plane case,

$$f_{10} = \ddot{H}\left(\frac{\zeta^4}{4H^2} - \frac{\zeta^5}{10H^3}\right) - \dot{H}^2\left(\frac{\zeta^4}{2H^3} - \frac{\zeta^6}{2H^5} + \frac{\zeta^7}{35H^6}\right) + \beta_1\zeta^3 + \beta_2\zeta^2,$$

and in the axisymmetric case

$$f_{10} = \ddot{H}\left(\frac{\zeta^4}{4H^2} - \frac{\zeta^5}{10H^3}\right) - \dot{H}^2\left(\frac{\zeta^4}{2H^3} - \frac{3\zeta^5}{10H^4} + \frac{\zeta^6}{10H^5} - \frac{\zeta^7}{35H^6}\right) + \beta_3\zeta^3 + \beta_4\zeta^2, \quad (33)$$

in which

$$\beta_1 = -\frac{\ddot{H}}{5H} + \frac{17\dot{H}^2}{35H^2}, \quad \beta_2 = \frac{\ddot{H}}{20} - \frac{9\dot{H}^2}{70H},$$

$$\beta_3 = -\frac{\ddot{H}}{5H} + \frac{5\dot{H}^2}{14H^2}, \quad \beta_4 = \frac{\ddot{H}}{20} - \frac{3\dot{H}^2}{35H}.$$

The dimensionless load parameter becomes in the plane case

$$w = -12\frac{\dot{H}}{H^3} + 12Wi\frac{\ddot{H}}{H^3} + Re\left(\frac{102}{35}\frac{\dot{H}^2}{H^2} - \frac{6}{5}\frac{\ddot{H}}{H}\right) + O(Re^2, Re\,Wi, Wi^2),$$

and in the axisymmetric case

$$w = -6\frac{\dot{H}}{H^3} - Wi\left(\frac{42}{5}\frac{\dot{H}^2}{H^4} - 6\frac{\ddot{H}}{H^3}\right) - Re\left(\frac{3\ddot{H}}{5H} - \frac{15\dot{H}^2}{14H^2}\right) + O(Re^2, Re\,Wi, Wi^2). \quad (34)$$

At zero Weissenberg number our axisymmetric perturbation solution (33) and (34) reduces to that of Kuzma (1967) and of Jones & Wilson (1975). It is to be noticed that, if both the velocity and acceleration of the top plate are negative, then inertia always increases the load whereas elasticity always decreases the load. Our latter conclusion agrees with those of Tanner (1965) and Kramer (1974); both of these authors ignored inertia contributions. Furthermore one notes that in both plane and axisymmetric cases the coefficients of the Weissenberg terms in the expressions for the dimensionless loads are much larger than the corresponding coefficients of the Reynolds terms. Thus one expects that elasticity soon dominates the flow and that our perturbation solution can only be valid for very small Weissenberg numbers. In contrast, in the corresponding Newtonian problem the load computed from the first-order perturbation theory shows good agreement with experimental results over a range of Reynolds numbers, extending to at least 60 (Kuzma 1967). In order to see when elasticity effects become important we note that the Reynolds number in a typical squeeze-film bearing is often greater than unity. Also $Wi = O(Re)$ if the relaxation time $\lambda \sim \rho h_0^2/\eta$. For a typical dilute solution where $\rho = 1000\ \mathrm{kg/m^3}$, $\eta = 0{\cdot}1$ Pa s and $h_0 = 1$ mm, it is found that $\lambda \approx 0{\cdot}01$. Thus a slightly non-Newtonian fluid may be expected to show dramatic departures from Newtonian behaviour in squeeze-film flow. Finally, because of the quasistatic nature of the perturbation solution we expect that it is valid only at large time, i.e. when τ is considerably greater than $(Re\,Wi)^{\frac{1}{2}}$; the latter timescale is the characteristic time for a linear transverse shear wave to travel one film thickness.

4. Numerical solution

To extend the perturbation solutions to any values of Wi and Re a numerical solution of the governing nonlinear equations is necessary. Before describing the numerical scheme we note that if the time τ and the space ζ are scaled with respect to Re (or Wi^{-1}) and the stresses are scaled accordingly ($X_1, R_1, Y, Z \to ReX_1, ReR_1, ReY, ReZ$; $T \to Re^2T$; $X_2, R_2 \to Re^3X_2, Re^3R_2$ and $\tau, \zeta \to Re^{-1}\tau, Re^{-1}\zeta$) then the governing equations are unchanged except that the equivalent Reynolds number is 1 and the Weissenberg number is $Wi\,Re$. Note that the boundary conditions (20) are preserved in this transformation. Thus as long as *Re* and *Wi* are not identically zero then we have the following results:

$$f(\zeta, \tau;\, Re, Wi) = f(Re\zeta,\, Re\tau;\, 1,\, Re\,Wi), \tag{35}$$

$$p(\tau;\, Re, Wi) = Re\, p\, (Re\tau;\, 1,\, Re\,Wi). \tag{36}$$

Equation (35) states that the evolution of the velocity field at a Weissenberg number *Wi* and at a Reynolds number *Re* is identical to that at a Reynolds number of unity and a Weissenberg number *Re Wi*. The time-scale in the second problem is $Re\tau$. This feature can be used to reduce numerical runs and checking that our codes work satisfactorily.

We employ an explicit numerical scheme which starts at the initial condition (31) and stops when the time τ reaches 0·6 (film thickness 0·4). First-order finite-difference formulae are used for time derivatives, and central-differences formulae are used for spatial derivatives. On the ζ-axis ($0 \leqslant \zeta \leqslant H(t)$) we select uniform nodal points at $\zeta_i = i\Delta h$, $i = 0, 1, \ldots, N$, where $\Delta h = H/N$. The variables are evaluated at time $\tau_j = j\Delta t$, where $j = 1, 2, \ldots$. At time $\tau = 0$ we impose the initial condition (31) and zero initial conditions for the stresses. We consider only the impulse-starting case where

$$H(\tau) = 1 - \tau, \quad \tau \geqslant 0+,$$

but our program can be easily modified to include any time-dependent $H(\tau)$. Denoting by f_j^n the nodal value at ζ_j and at time τ_n, we have for the plane case, from (19),

$$\sum_{j=1}^{N-1} A_{ij} f_j^n = \Delta h^2 \alpha_i^n + f_N^n\, \delta_{i,\,N-1} \quad (i = 1, \ldots, N-1),$$

where A_{ij} is the Rouse $[N-1,\, N-1]$ matrix

$$A_{ij} = \begin{cases} 2 & (i = j), \\ -1 & (i = j \pm 1), \end{cases}$$

and α_i^n is the expression $\quad \dfrac{1}{Re}(2X_2' + T'') - f'f'' + ff'''$

evaluated at node ζ_i and at time τ_n.

Second-order finite-difference formulae for spatial derivatives are employed:

$$f_i'^{\,n} = \frac{f_{i+1}^n - f_{i-1}^n}{2\Delta h},$$

$$f_i''^{\,n} = \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{\Delta h^2},$$

$$f_i'''^{\,n} = \frac{\tfrac{1}{2}f_{i+4}^n - f_{i+3}^n + f_{i+1}^n - \tfrac{1}{2}f_i^n}{\Delta h^3}.$$

To account for boundary conditions (20) we invent fictitious nodes at $\zeta_{-1} = -\Delta h$ and at $\zeta_{N+1} = (N+1)\Delta h$ such that

$$f_{-1}^n = f_1^n \quad (f'(0) = 0),$$

$$f_{N+1}^n = f_{N-1}^n \quad (f'(H) = 0).$$

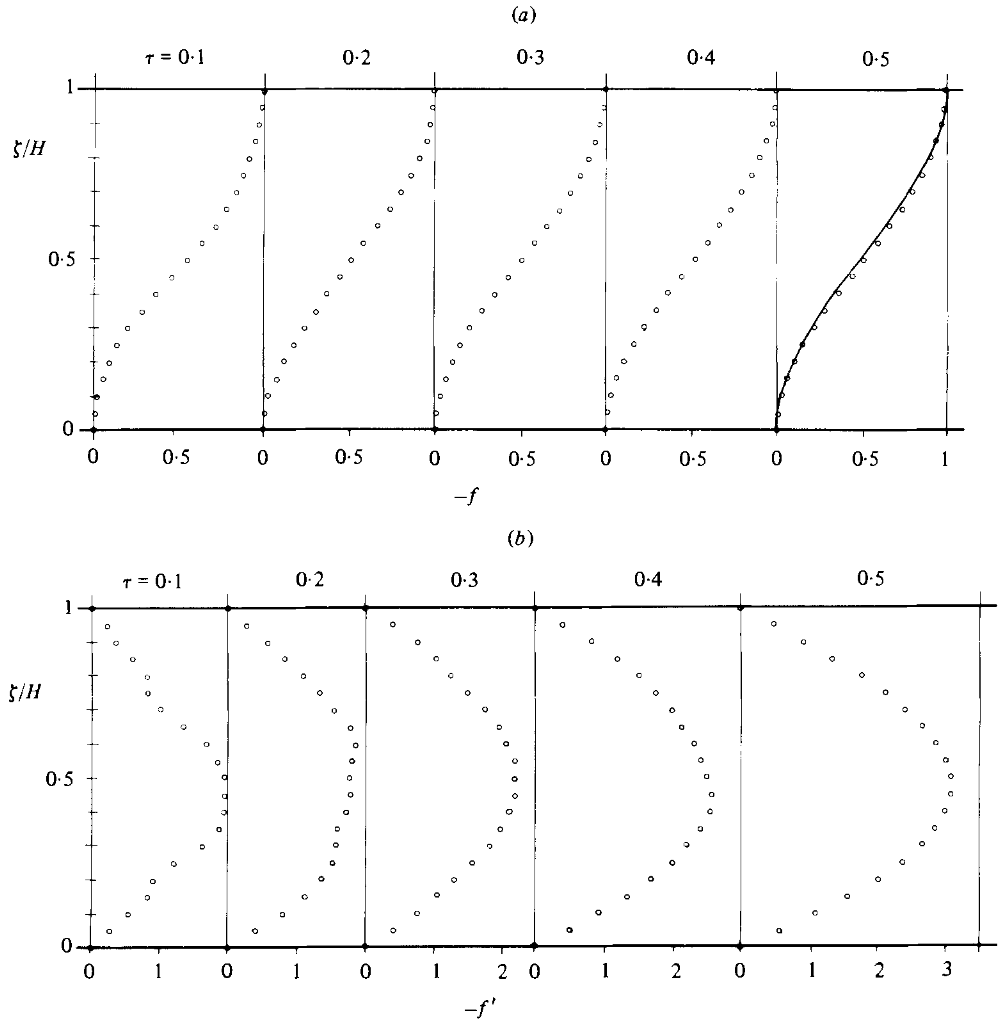

FIGURE 2. Vertical $(-f)$ and horizontal velocity $(-f')$ at $Re = 1$ and $Wi = 0{\cdot}1$. The solid line in (*a*) is the classical lubrication approximation. Note the radial plug flow in the central region and that f' is not symmetric about $\zeta = \frac{1}{2}H$.

At $\zeta = 0, f = f_0^n = 0$, and, at $\zeta = H, f = f_N^n = \dot{H} = -1$. Since the inverse of the Rouse matrix is the Kramers $[N-1,\, N-1]$ matrix

$$C_{ij} = \begin{cases} i(N-j)/N & (i \leqslant j), \\ j(N-1)/N & (j \leqslant i), \end{cases}$$

we have for the interior nodes ζ_i, $i = 1, \ldots, N-1$,

$$f_i^{n+1} = f_i^n + \Delta t \sum_{j=1}^{N-1} C_{ij}(\Delta h^2 \alpha_j^n + \ddot{H}\delta_{j,\,N-1}).$$

The constitutive equations can be all put in the form

$$\dot{\psi} + \frac{1}{Wi}\psi = g.$$

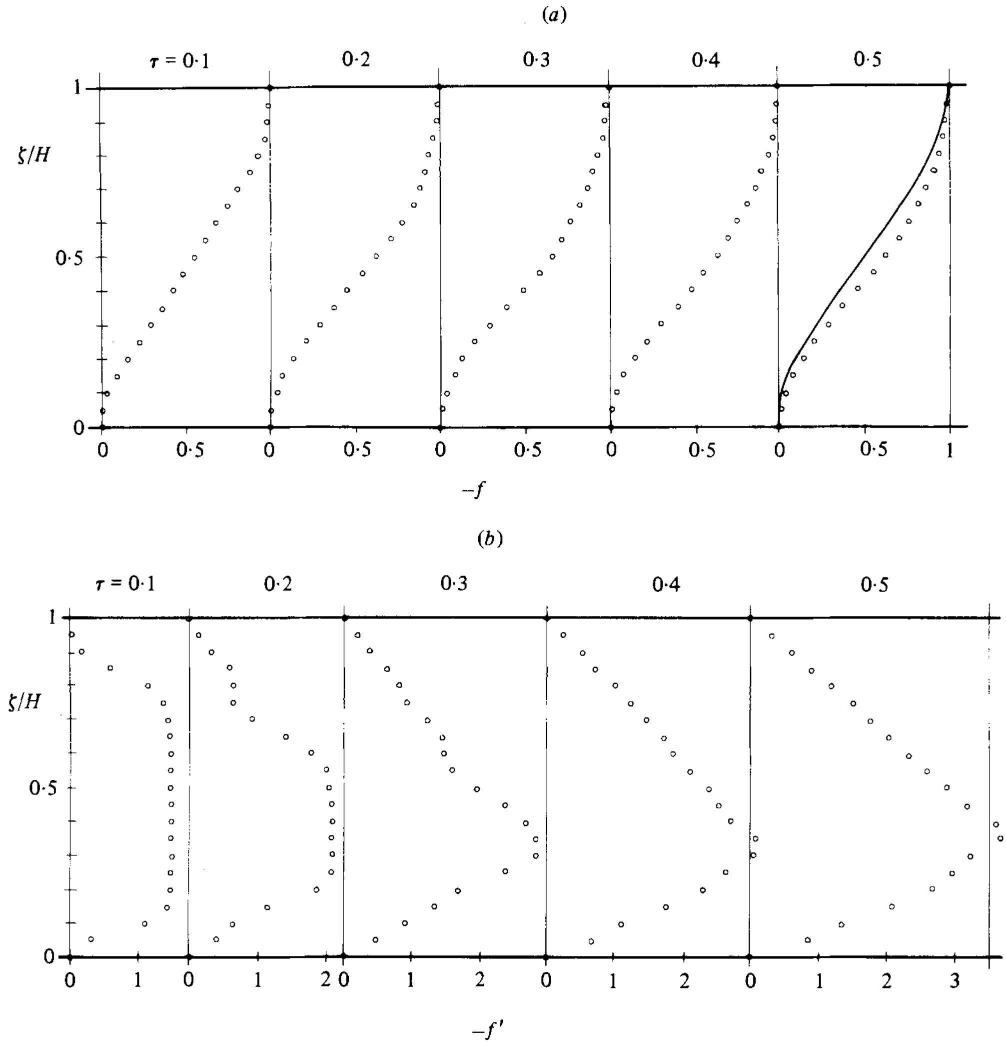

FIGURE 3. Same as in figure 2, but with $Wi = 1$. Note the transverse shear wave spreading in from the top plate and the established triangular horizontal velocity in (*b*).

They are updated using

$$\psi_i^{n+1} = \psi_i^n \exp\left(\frac{\Delta t}{Wi}\right) + Wi\, g_i^n \left(1 - \exp\left(-\frac{\Delta t}{Wi}\right)\right).$$

To update the stresses at the boundary node where $\zeta_N = H(t)$, we use the first-order finite-difference formula for the spatial derivatives of the stresses. After all the variables have been updated at time τ_{n+1}, a new film thickness is computed and a set of nodal points are selected (keeping the number of intervals N constant). The procedure is repeated until $H = 0{\cdot}4$. The calculations were done on a computer that retains 15 significant figures. The codes are tested by reducing both Wi and Re to near zero. We found that, up to Wi and Re of order 0·1, our numerical results are indistinguishable from the first-order perturbation solutions. In figures 1–7 we display the axisymmetric velocity fields at different values of Re and Wi. In all cases we found that the similarity conditions (35) and (36) are valid to at least 4 significant figures

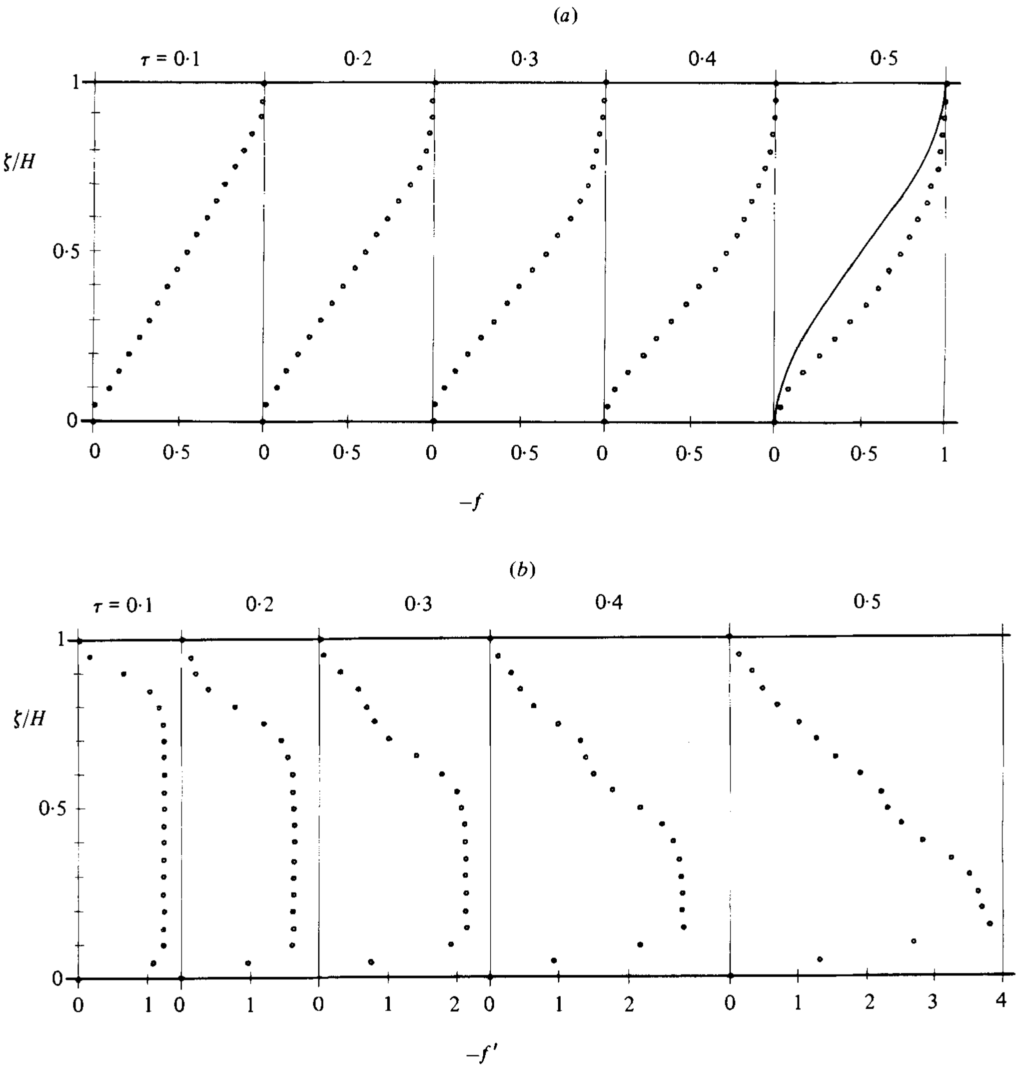

FIGURE 4. Same as in figure 2, but with $Wi = 10$. Note the thin boundary layer near the bottom plate.

(we only printed out four significant figures). Furthermore, the plane and axisymmetric results are quite similar, and consequently we display only axisymmetric results. It is noteworthy that a straightforward evaluation of the loads using (18), (27) and finite-difference formulae is very inaccurate. A better way is to compute the spatial averages of (18) and (27):

in the plane case

$$p(\tau) = \frac{2}{H}\int_0^H (X_2 - Ref'^2)\,d\zeta + \frac{T_N - T_0}{H} + Re\frac{\ddot{H}}{H},$$

and in the axisymmetric case

$$p(\tau) = \frac{3}{H}\int_0^H (R_2 - \tfrac{1}{4}Ref'^2)\,d\zeta + \frac{T_N - T_0}{H} + \tfrac{1}{2}Re\frac{\ddot{H}}{H}.$$

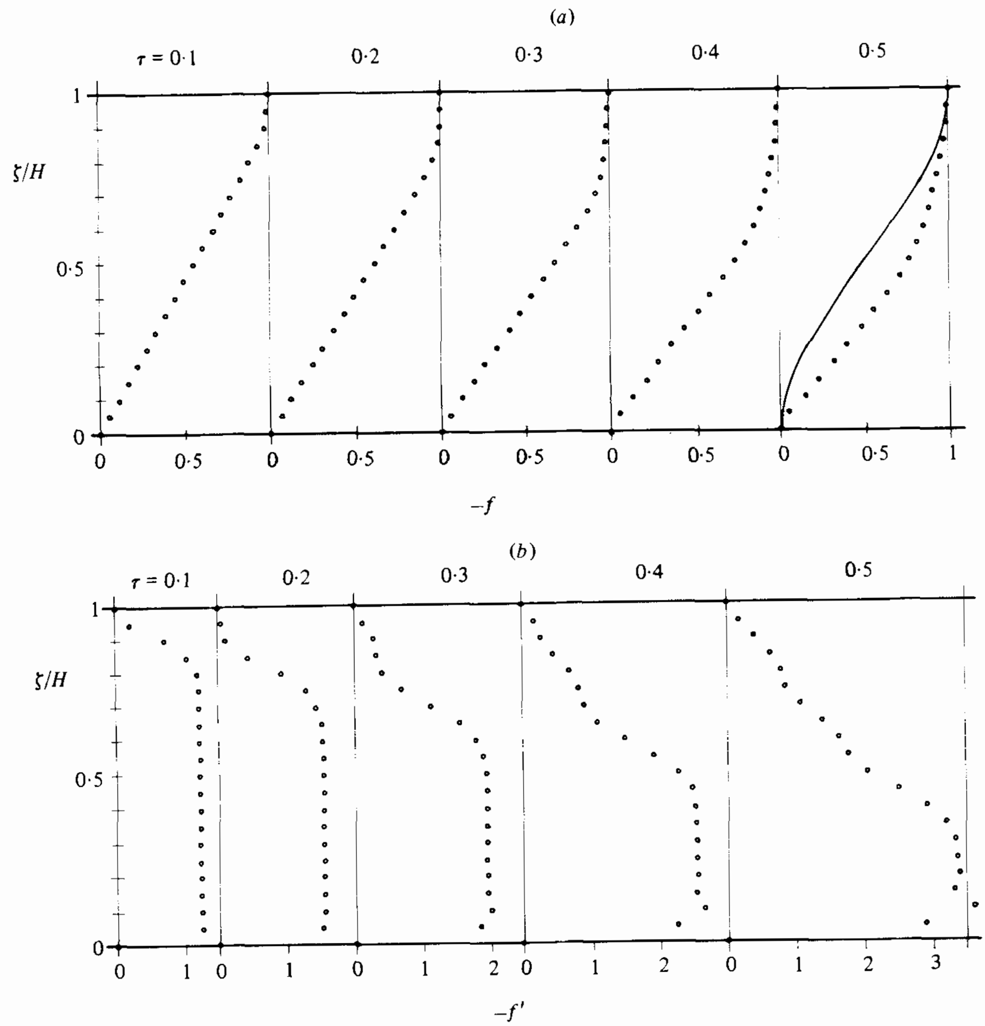

FIGURE 5. Same as in figure 2, but with $Wi = 50$.

The integrations were carried out by the trapezoidal rule. Hamza & MacDonald (1981) used the same method to evaluate their loads, but with a weighting function $(1-\zeta/H)\,\zeta/H$ instead of a uniform weighting function of unity.

The effects of elasticity can be gauged from figures 2–7, where we display the vertical velocity $-w/V = -f$ and the horizontal velocity $u/\frac{1}{2}rV = -f'$ at a Reynolds number of 1 and at different Weissenberg numbers extending to 500. It is noticed that there is unavoidable numerical noise in f' (and hence in u), but the trend recorded in these figures is believed to be real. For small times we see a radial plug flow in the centre region of the bearing, and two boundary layers adjacent to the plates. At high $Re\,Wi$ the boundary layer adjacent to the bottom plate is extremely thin compared with that adjacent to the top plate. Of interest is the load parameter w, which we have plotted in figure 8. Clearly at all times $\tau > 0$ an increase in the Weissenberg number (keeping Re constant) results in a decrease in w and thus the load-carrying capacity of the bearing. This conclusion agrees with Tanner (1965), who used a lubrication approach, and with Kramer (1974); both these authors neglected inertia.

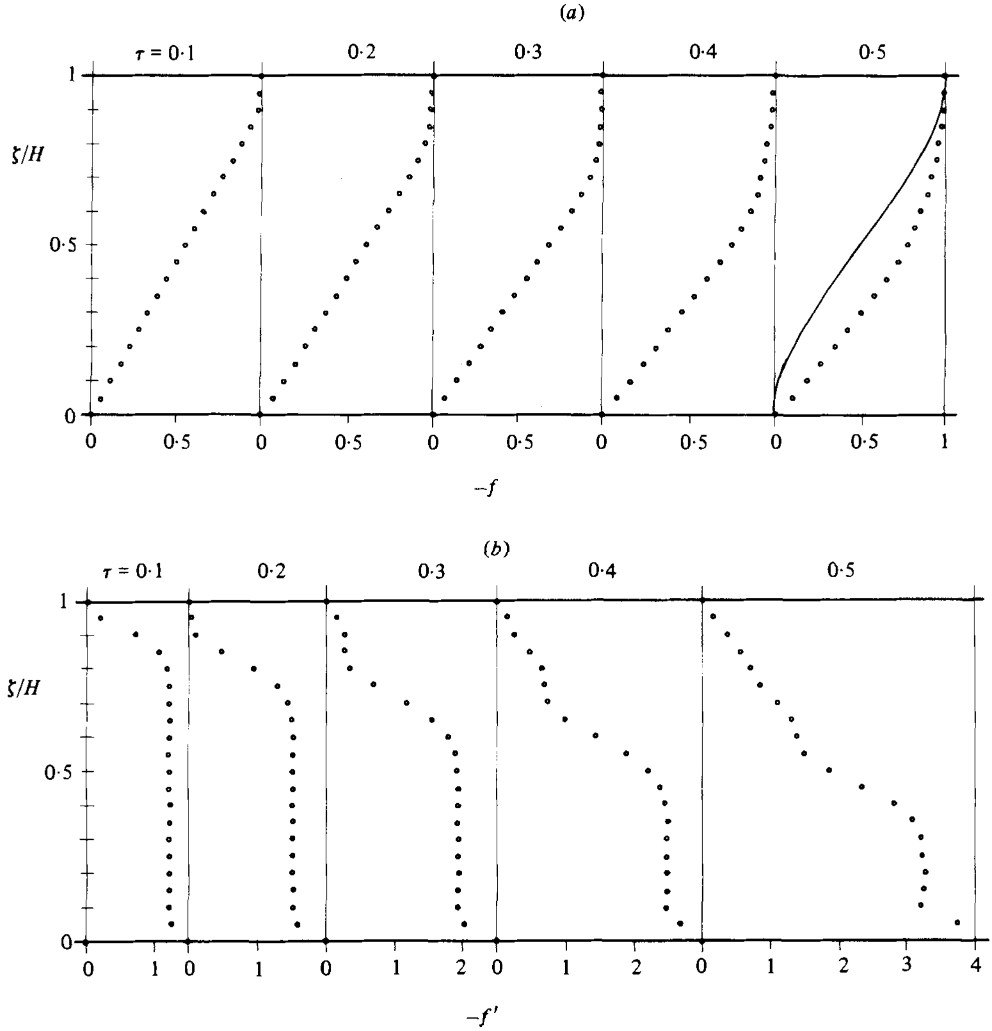

FIGURE 6. Same as in figure 2, but with $Wi = 100$.

5. Conclusions

It seems remarkable that an unsteady, fully two-dimensional solution has been obtained to the problem in hand at very large Weissenberg numbers, especially in view of the problems encountered in the numerical solution of viscoelastic flow problems at medium Weissenberg numbers by finite-element and finite-difference methods. The reason that numerical instability is absent here is not known, but may be due to the constrained forms assumed for the variables, which avoid the need to consider the incompressibility constraint and the pressure field explicitly.

There is no need to let fluid planes 'buckle' in the present problem: the solution is exact. The nature of the solution changes from the simple parabolic velocity distribution in low-Weissenberg-number creeping flow (fig. 2*b*) to a form (e.g. figure 6*b*) in which transverse shear waves propagate across the gap. We are not aware that these have been observed previously in numerical solutions of viscoelastic flows; they are expected as shown by the solution to the suddenly accelerated plate (Rayleigh)

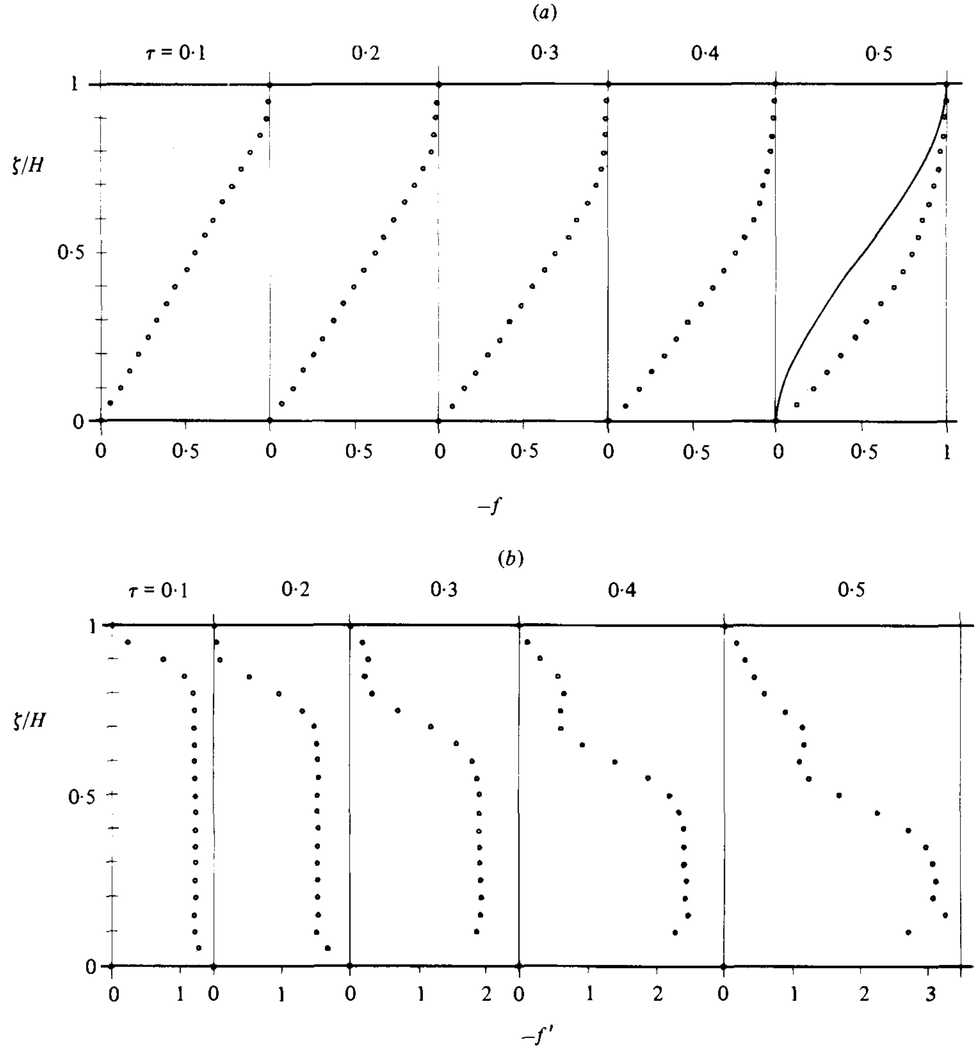

FIGURE 7. Same as in figure 2, but with $Wi = 500$.

problem (Tanner 1962). In terms of the dimensionless variables used in the present paper the transverse wave speed is $(Re\ Wi)^{-\frac{1}{2}}$ from the linear analysis of Tanner (1962). However, the fluid moves downwards at each point, and so the speed of propagation of a disturbance across the gap is better described (in dimensionless terms) as $(Wi\ Re)^{-\frac{1}{2}}+f$. This agrees well with the numerical solutions; at very high Weissenberg–Reynolds numbers a disturbance is practically convected with the fluid.

The edge effects which have been treated by setting the total force on the exit edge to zero need to be discussed. Some disturbance to the flow inside the gap must be expected if the edge effect was taken into account in a strict manner. At medium products of *Re Wi* one can argue that the flow is equivalent to a boundary-layer flow (high *Re*, low *Wi*) and the situation is similar to the flow leaving the rear edge of a wing or flat plate. In such cases the disturbance to the pressure field is negligible and thus we believe the solution is applicable to discussion of the load-bearing capacity of finite plates, provided that the mean-zero-force boundary condition is applied.

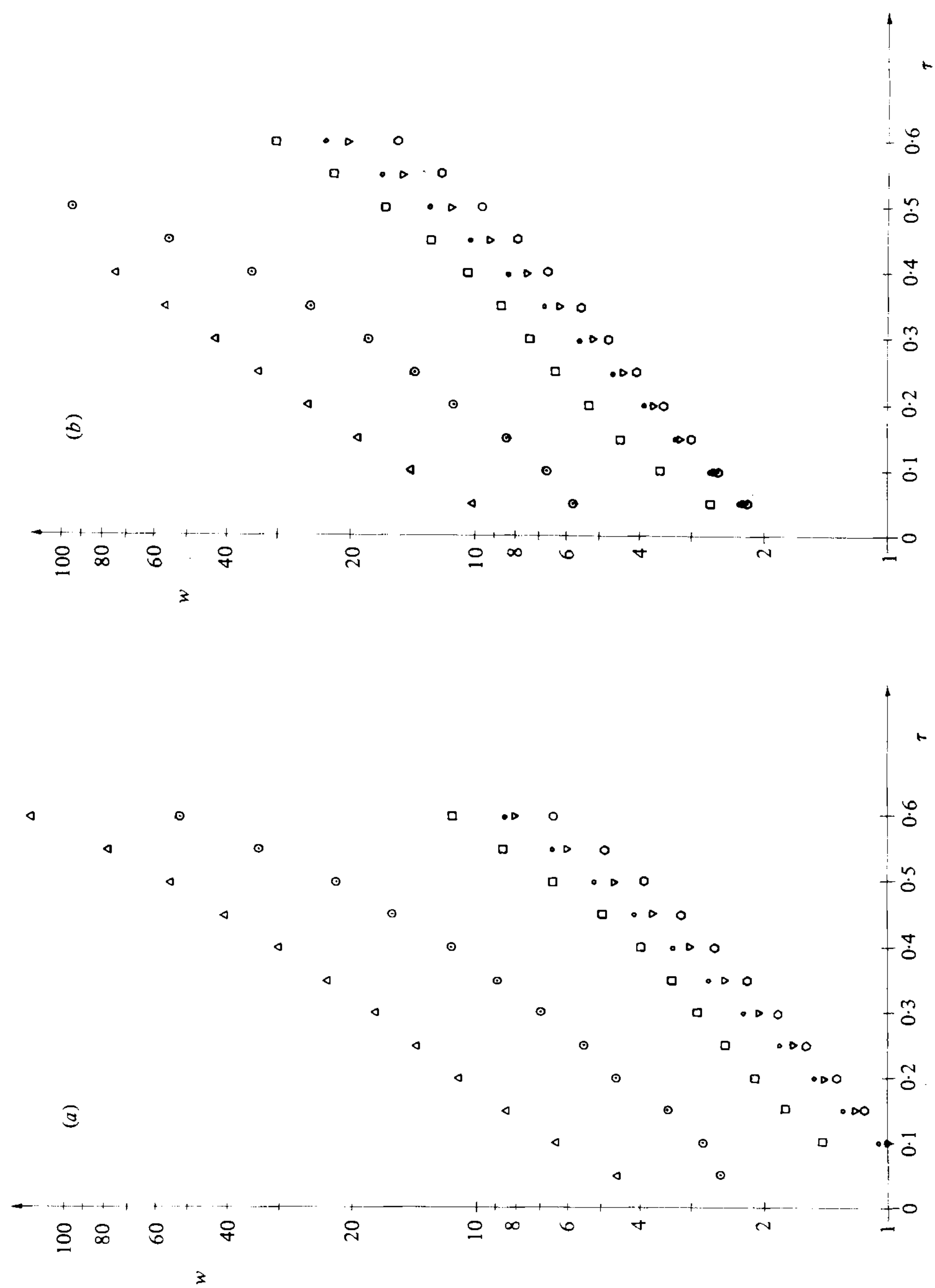

FIGURE 8. Axisymmetric and plane dimensionless load parameters w ((30) and (29) respectively) as functions of time. △, $Wi = 0{\cdot}1$; ⊙, 1; □, 10; ○, 50; ▽, 100; ⬡, 500. In all cases $Re = 1$.

The solution shows a reduced load capacity relative to the Newtonian case, thus agreeing with previous less complete calculations. The experiments of Grimm (1975) show, however, an increased capacity relative to the shear-thinning inelastic computation. Two ideas that have been put forward to explain this difference viscoelasticity and the increased resistance to stretching flow (Metzner 1968) have been incorporated in the present solution and are therefore not thought to be very relevant. The remaining idea is that 'stiffer' films result from the overshoot of shear stress when a sudden shear rate is applied to a viscoelastic sample (Bird, Armstrong & Hassager, 1977). The Maxwell fluid used here does not possess this property, and further exploration of this idea needs to be made with another constitutive model which includes this effect and shear thinning.

We gratefully acknowledge support for this study from the Australian Research Grants Scheme.

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